Ján GUNČAGA, Štefan TKAČIK, Ružomberok, Slovakia

Historical remarks to integration methods[1]

This article makes an excursion into the history of integration, from ancient to the present time. Our presentation is based on the exhaustion method invented by Eudoxos and used by Archimedes and Oresme in their integration attempts. The aim of our article is to show that this approach leads to an integral equivalent to the Lebesgue integral. It is based on the summation of infinite series and it has some pedagogical advantages. A weaker form of it completely avoids measure theory and it is simpler than the Riemann integral.

1. Archimedes method for the finding the area of a parabolic section

The proof is one of reductio ad absurdum, and the method is to show that, if the diagonal of a square is commensurable with the side, then the same number must be both odd and even. The understanding that the diagonal of a square is not commensurable with its side lead to the better understanding of numbers: the notion of cardinality in terms of natural (and rational) numbers is not rich enough to express various forms of the length. The calculations by Archimedes provided strong impetus for the development of real number. The method of finding the area of parabolic section as he presented it so far is surely not rigorous by our standards. Archimedes did not stop at the picture, he offered a fine argument. His argument became an important principle of mathematical analysis. He found this important property by calculating the sum of the areas of triangles, which fill parabolic section. We have to calculate the following sum: $A+\frac{1}{4}A+\frac{1}{4^2}A+\frac{1}{4^3}A+\dots$ He proceeded in an ingenious way illustrated in Figure 1. From a square one corner is removed. The removed corner is a square whose side is half of that of the original square and, hence, its area is $\frac{1}{4}$ of the area of the original square. The resulting area of three squares equals the area of A, so that the area of the original square is $\frac{4}{3}A$. The same is done with the removed square in the

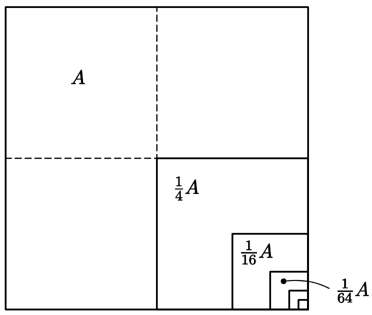

Figure 1

[1] **Remark:** Supported by Grants KEGA Nr. 3/7068/09, Nr.168-010KU-4/2010.

corner, then with the square removed from this square and so on. The union of all figures so obtained is the whole of the original square. Therefore $A+\frac{1}{4}A+\frac{1}{4^2}A+\frac{1}{4^3}A+...=\frac{4}{3}A.$ Archimedes proved, that $\frac{4}{3}A$ is the smallest of all numbers B such that $A+\frac{1}{4}A+\frac{1}{4^2}A+\frac{1}{4^3}A+...+\frac{1}{4^n}A<B, n=0,1,2,...$

Archimedes needed for this purpose the following property of numbers which have now name the principle of Archimedes:

For any numbers $\varepsilon > 0$ and K, there exists $n \in N$ such that $\frac{1}{n}K < \varepsilon$.

By this way we introduce the sum of a sequence of positive numbers.

Archimedean definition. *A number s is called the sum of the sequence of positive numbers* $a_0, a_1, a_2,\ldots$ *if s is the smallest of all the numbers b such that* $a_0 + a_1 + a_2+\ldots + a_n \leq b$ *for every* $n = 0, 1, 2,\ldots$

If s is the sum of the sequence of numbers $a_0, a_1, a_2,\ldots$, *then we write*

$a_0 + a_1 + a_2+\ldots = s.$

This approach to infinite sums is easier and avoids limits. The correct definition of the limit of a sequence requires the involvement of three quantifiers. On the other hand, in the Archimedean definition of sum we got away with only 2 quantifiers.

2. "Archimedes" integral - application of the Archimedean methods

Nicole d'Oresme around 1350 has shown that $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+...=2.$ The sum was obtained by modifying the geometric method of Archimedes. He divided the same planar region into rectangles in 2 ways. The first way, rectangles have base of length $\frac{1}{2^n}$ and height n, therefore its area is equal $\frac{n}{2^n}$ and the sum of areas of these rectangles is $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+...$ The second way, each rectangle has base of length $\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+...=\frac{1}{2^n}$. Hence

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+...=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...=2.$$

The determination of area of a parabolic section by Archimedes on Figure 2 and the finding sum by Oresme are both based on the following property of area which is called σ-additivity: If a planar set S is equal to the union of sets S_1, S_2, S_3,… such that the common part of any two of them has the area equal to zero, then the area S is equal to the sum of the areas of the sets S_1, S_2, S_3, … Archimedes and Oresme´s calculations show that this property can be used for determination of areas of planar sets and also for finding sums of sequences. If we interpret the integral of a positive function as the area of "region under its graph", then we can of course use this property for finding integrals.

Let us translate these geometric ideas into analytic language. The length of bounded interval I, that is, the absolute value of the difference of its end-points, is denoted by $\lambda(I)$. Characteristics function of the interval I, we denoted by χ_I, so $\chi_I(x) = \begin{cases} 1 & x \in I \\ 0 & x \notin I \end{cases}$. If c is an positive number and I a bounded interval, then the „region under the graph" of the function $c\chi_I$ is a rectangle whose base has the length $\lambda(I)$, the height is c and the area is $c\lambda(I)$. Using these conventions we can define "Oresme" integral.

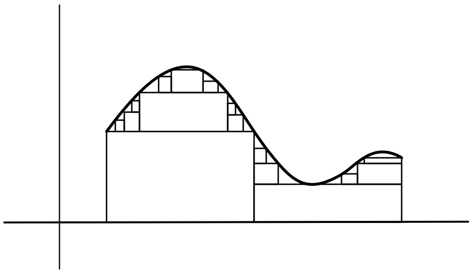

Figure 2

Definition 1. Lets function f is a nonnegative in the interval I. Let c_i nonnegative numbers and intervals $I_i \subseteq I$, i = 1, 2, 3, … such that following condition is satisfy :

$f(x) = \sum_{i=1}^{\infty} c_i \chi_{I_i}(x)$ for every $x \in I$ and exist the sum $\sum_{i=1}^{\infty} c_i \lambda(I_i)$.

Oresme integral in the interval I is then defined (O)$\int_I f\, d\lambda = \sum_{i=1}^{\infty} c_i \lambda(I_i)$.

This definition is valid for the nonnegative and bounded functions (see Figure 2). Although the class of such functions is already quide wide, Oresme´s derivation suggest that it can be widened. For example, it is possible to include some functions with negative values. When we cover the "region under the graph" of function, we do not have to stay with the rectangles strictly within that region. If we overshoot with some, we subtract the areas of rectangles covering the excess. In figure 3 the subtracted rectangles are shaded. In this case we need following modified definition.

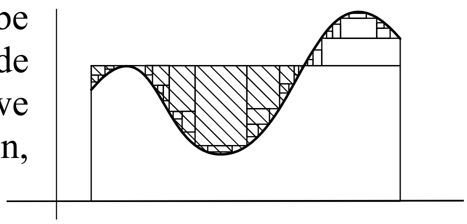

Figure 3

Definition 2 Lets function f is define in the interval I. Let c_i are numbers and intervals $I_i \subseteq I$, i = 1, 2, 3, … such that following condition is satisfy:

1. $f(x)=\sum_{i=1}^{\infty}c_i\chi_{I_i}(x)$ for every $x\in I$,

2. $\sum_{i=1}^{\infty}|c_i|\chi_{I_i}(x)<\infty$ for every $x\in I$,

3. $\sum_{i=1}^{\infty}|c_i|\lambda(I_i)<\infty$

Kluvánek integral in the interval I is then defined $(\mathrm{K})\int_I f\,d\lambda=\sum_{i=1}^{\infty}c_i\lambda(I_i)$.

For example, it is possible to include some unbounded functions and it is not necessary to assume that the underlying interval I be bounded. But of course, some precautions then have to be taken because the area of the "region under the graph" may be infinite and it may not be possible or easy to cover the "region under the graph" by rectangles without also covering some points off that region. In this case we need following modified definition.

Definition 3.
Lets function f is define in the interval I. Let c_i are numbers and intervals $I_i\subseteq I$, $i = 1, 2, 3, \ldots$ such that following condition is satisfy :

1. $f(x)=\sum_{i=1}^{\infty}c_i\chi_{I_i}(x)$ for every $x\in I$, for which hold $\sum_{i=1}^{\infty}|c_i|\chi_{I_i}(x)<\infty$,

2. $\sum_{i=1}^{\infty}|c_i|\lambda(I_i)<\infty$.

Archimedes integral in the interval I is then defined $(\mathrm{A})\int_I f\,d\lambda=\sum_{i=1}^{\infty}c_i\lambda(I_i)$.

3. Conclusions.

The last question in our article is about the relation of Archimedes integral to other notions of integrability and integral found in the literature. It turns out that a function is Archimedes integrable if and only if it is Lebesgue integrable and the Archimedes integral of such function coincides with its Lebesgue integral. The approach to integration outlined in our article has a strong historical aspect and presents integration in a simple and effective way, hence it is suitable from educational point of view.

References

[1] Kluvánek, I. (1987): Archimedes was right, Elemente der Mathematik, Vol. 42, Nr.4, 93-114.

[2] Tkačik, Š. (2010): Historické poznámky o metódach integrovania, Scientific Issues, Teaching Mathematics II, Ružomberok, 179-190. ISBN 978-80-8084-645-9